The “frac_decomp” program : decomposition of a rational fraction, a linear system, into simple elements for real polynomials

1 Introduction

This function can replace the Scilab macro `pfss.sci`. The proposed method (algebraic) is not the same as the method used by Scilab ; here we use the `mroots.sci` program which gives, for a real polynomial, the roots and the corresponding multiplicities.

We start by looking for the elementary polynomials, of the first and second degree and the corresponding multiplicities. Depending on these polynomials and the multiplicities, we calculate by solving a system of linear equations, the residue relating to each of the roots, the rest of the program consists of a presentation in vector form, the simple elements (we also give the polynomial quotient of the two polynomials numerator and denominator if the degree of the numerator $>=$ the degree of the denominator). The syntax of this function is: `[result] = frac_decomp(g,flag)`

2 Principle of the algorithm

We form, a priori, the expression of the decomposition by carrying out the Euclidean division of the numerator Num by the denominator Den of the starting fraction r, i.e. `[rest,elm1] = pdiv(Num,Den)`. The degree of polynomial `elm1` is : degree(Num)-degree(Den), if `degree(Num) >= degree(Den)`. Once this is done, we put the remaining fraction into simple elements : **G** $=$ **Num/Den - elm1**. To carry out this decomposition, we calculate the roots and the multiplicities of the roots of `Den` with the program `mroots.sci` : this is mandatory. It is with this program that we characterize the roots in two categories : real roots and imaginary roots, taking into account the multiplicities, we can thus determine the simple elements and construct a matrix of polynomials A. By identifying the degrees of the two members of the polynomial equations we solve a system of linear equations whose solution gives the values of the coefficients of the decomposition : $C = (A^t)^{-1}COE$, The vector COE being the coefficients of the numerator of G. Here is a simple example :

```
--> r=(x^9+2*x)/((x+6)*(x+1)^3*(x^2+x+7)*(x^2+x+1)^2)
```

```
r =

                               2x +x^9
   -----------------------------------------------------------------
   42 +223x +576x^2 +948x^3 +1079x^4 +876x^5 +510x^6 +209x^7 +60x^8 +12x^9 +x^10
   --> Num = r.num ; Den=r.den ;
   --> rm=mroots(Den)
rm =
        -6.  + 0.i            1.  + 0.i //single real root
        -1.  + 0.i            3.  + 0.i //real triple root
        -0.5 + 2.5980762i    1.  + 0.i //single imaginary root
        -0.5 - 2.5980762i    1.  + 0.i //Racine imaginaire conjuguée
simple
        -0.5 + 0.8660254i    2.  + 0.i //Racine imaginaire double
        -0.5 - 0.8660254i    2.  + 0.i //Racine imaginaire conjuguée
double
```

We construct the elementary polynomials of first and second order in the order of appearance of the roots, namely :

2.0.1 The whole part, if necessary

We carry out, after having made the polynomial Den unitary if this is useful, the Euclidean division of Num by Den is :

```
--> [rest, elm1] = pdiv(Num,Den] ; G = rest/Den ;
```

We will use the two results found at the end of the program

2.0.2 Elementary real polynomials.

polr =
6 +x //single root
1 +x //triple root
1 +2x +x² //(1+x)^2
1 +3x +3x² +x³//(1+x)^3

For a single real root we will have a single element of the form: $c_1 \frac{1}{6+x}$. Then, for the triple root we have the elements : $c_2 \frac{1}{1+x}$ then $c_3 \frac{1}{(1+x)^2}$ and finally $c_4 \frac{1}{(1+x)^3}$.(The c_i are constants to be determined).

2.0.3 Imaginary roots (conjugated)

We finally construct the trinomials with the conjugated imaginary roots then the corresponding elementary polynomials here is an example with two trinomials, one $(2 + 2s + s^2)$ is simple and $(2 + s + s^2)$ has a multiplicity 2 .

New example :

```
   --> s=%s; g=syslin("c",(3+s)/(s*(2+2*s+s*s)*(2+s+s*s)**2))
   --> decomp=frac_decomp(g);
decomp =
```

```
0 0.375  -1.125   -0.125s        -1.25              0.25s          0.5     -0.25s
- ----- --------- --------- ------------------ ------------------ --------- ---------
1   s    2+1s+s²   2+1s+s²   4 +4s +5s²+2s³+s⁴   4 +4s+5s²+2s³+s⁴  2+2s+s²   2+2s+s²
```

We can clearly see the two trinomials appear, the simple one with the two fractions $\frac{0.5}{2+2s+s^2}$ and $\frac{-0.25s}{2+2s+s^2}$; then the trinomial of multiplicity 2 with four fractions $\frac{-1.125}{(2+s+s^2}$ and $\frac{-0.125s}{2+s+s^2}$ then, $\frac{-1.25}{(2+s+s^2)^2}$ and $\frac{0.25s}{(2+s+s^2)^2}$. In practice we recombine the two fractions to give the numerator a polynomial of degree 1 : `flag="y..."` in the program.

A little check:

```
--> clean(sum(decomp),0,1.e-14)
ans =
3 +1s
-----------------------------------
8s +16s² +22s³ +18s⁴ +11s⁵ +4s⁶ +s⁷
```

Now let's use the Scilab pfss.sci program, and compare the results.

```
   --> elts=pfss(r,1.e+15)
elts =
   (1) :  [1x1 rational] of s
(2) :  [1x1 rational] of s
(3) :  [1x1 rational] of s
(4) :  [1x1 rational] of s
   -->velts = list2vec(elts)'
   0 -0.1117919 -0.1055645s 0.0867818 -0.0160205s 0.1577441
   - --------------------- --------------------- ------------
   1    2 +1.0000001s +s²           2 +2s +s²      3.048D-16 +s
--> sum(velts)//??????
```

2.0.4 Construction of the linear system to be inverted

Once we set aside the entire part of the ratio of the two polynomials Num/Den, we group the simple elements into two subsets:

The simple elements corresponding to the simple or multiple real poles of the denominator.

The simple elements corresponding to the imaginary single or multiple conjugate poles of the denominator.

Indeed we can write the relation $\frac{Num}{Den} = Partieentiere + \sum realelems + \sum imagelems)$; or $rest = Den(\sum realelems) + Den(\sum imagelems)$

The right part of this equation can be put in the form: $(c_1, c_2, c_3, \cdots, c_i, \cdots)* (Den/(x+a_1), Den/(x+a_2)^2 \ldots) \cdots (Den + x * Den)/(x^2 + b_i x + d_j)^k)^t$

This system of polynomial equations is linear with respect to the parameters c_j, we can therefore invert the matrice of the coefficients of the polynomials and take into account the left part of the equation which is also a determinable vector COE.