Article + Errata

Curve Fitting for Ogden, Yeoh and Polynomial Models

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Contents

1 Article

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This document provides the theoretical background for the SCILAB curve fitting scripts for Ogden, Yeoh and Polynomial models, as presented at ScilabTEC conference 2015.

Please cite [Rackl (2015)], if you find these information useful.

Contents

1. Hyperelastic Material Models

1.1. Ogden Model

The Ogden model was developed by Ogden (1972). It expresses the strain energy function W in terms of principal stretches λ_1, λ_2 and λ_3. The formulation is shown in Equation 1, where μ_p and α_p are material constants.

$$W = \sum_{p=1}^{n} \frac{\mu_p}{\alpha_p} \cdot \left(\lambda_1^{\alpha_p} + \lambda_2^{\alpha_p} + \lambda_3^{\alpha_p} - 3\right) \tag{1}$$

1.2. Polynomial Model

The polynomial hyperelastic model was introduced by Rivlin & Saunders (1951). It is formulated in terms of the two strain invariants I_1 and I_2 of the left Cauchy-Green deformation tensor. With C_{ij} denoting material constants, its strain energy is

$$W = \sum_{i=0,j=0}^{n} C_{ij} \cdot \left(I_1 - 3\right)^i \cdot \left(I_2 - 3\right)^j \quad \text{, where } C_{00} = 0. \tag{2}$$

This model is also called the *generalized Rivlin model* (Chang et al. 1991, Hartmann & Neff 2003, Laksari et al. 2012).

A specific case of Equation 2 was developed earlier by Mooney (1940) and Rivlin (1948*b*). It is referred to as the *Mooney-Rivlin model* and can be derived from Equation 2 by setting $n = 1, C_{01} = C_2, C_{11} = 0$ and $C_{10} = C_1$, which yields Equation 3 for the strain energy W.

$$W = C_1 \cdot \left(I_1 - 3\right) + C_2 \cdot \left(I_2 - 3\right) \tag{3}$$

1.3. Yeoh Model

Yeoh (1993) developed a hyperelastic material model that only depends on the first strain invariant. Like the Ogden and polynomial model, it is also based on a series expansion. In the original article, Yeoh truncated the series after the first three terms. However, a more general definition is used nowadays (Selvadurai 2006). Its definition of the strain energy is

$$W = \sum_{i=1}^{n} C_i \cdot \left(I_1 - 3\right)^i. \tag{4}$$

2. Curve Fitting Using the Example of a 5 Parameter Polynomial Model

Hyperelastic material model fitting "is a very delicate" issue (Ogden et al. 2004).

The aim of curve fitting is to fit the parameters of a model function in such a way that the fitted curve is as close to the measured curve as possible. In order for the fitted model to represent the real behavior of the material, more than one loading mode should be considered. In addition to that, the selected loading modes shall be similar to the desired loading case of the application (Meier et al. 2003).

Curve fitting for hyperelastic material models divides into four subtasks. First of all, the stress-strain curves need to be revised and adapted, where required. Second, a constitutive equation (material model) has to be chosen, and third, an error criterion for the goodness of fit has to be defined. The final step is to qualitatively and quantitatively compare the resulting curve to the measured data.

The curve fitting process is similar for the Yeoh and Ogden hyperelastic models. The stress-strain relationships for both models are derived in the appendix (subsection A.1).

2.1. Derivation of the Stress-Strain Relationships for a 5 Parameter Polynomial Model

The curve fitting process is the same for different hyperelastic models. Stress-strain relationships for the same loading modes, in which the stress-strain curves were measured, have to be derived. In the following, the stress-strain relations of a 5 parameter polynomial model will be derived in compression/tension and simple shear.

Recalling Equation 2 from section 1, the constitutive equation of the polynomial model defines the strain energy W as (ANSYS Inc. n.d.)

$$W = C_{10}\cdot\left(I_1-3\right)+C_{01}\cdot\left(I_2-3\right)+C_{20}\cdot\left(I_1-3\right)^2+C_{11}\cdot\left(I_1-3\right)\cdot\left(I_2-3\right)+C_{02}\cdot\left(I_2-3\right)^2. \quad (5)$$

The strain invariants, expressed in terms of the three stretch ratios λ_1, λ_2 and λ_3, are (Treloar 1973)

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^3, \quad (6)$$
$$I_2 = \lambda_1^2 \cdot \lambda_2^2 + \lambda_2^2 \cdot \lambda_3^2 + \lambda_3^2 \cdot \lambda_1^2 \qquad \text{and} \quad (7)$$
$$I_3 = \lambda_1^2 \cdot \lambda_2^2 \cdot \lambda_3^2. \quad (8)$$

The stretch ratios λ_i represent the deformation of a differential cubic volume element along the principle axes of a Cartesian coordinate system. They are defined as the ratio of the deformed length l_i to the undeformed length L_i (Equation 9). The stretch ratio equals 1 in undeformed state.

$$\lambda_i = \frac{l_i}{L_i} \qquad i \in [1, 2, 3] \quad (9)$$

Note that given the assumption of incompressibility of the material, the third strain invariant I_3 yields

$$I_3 = \lambda_1^2 \cdot \lambda_2^2 \cdot \lambda_3^2 = 1, \quad (10)$$

because of the conservation of the volume of a differential volume element. As a result, the incompressible polynomial model is expressed in terms of the first and second strain invariant only.

Combining Equation 6 and Equation 7 with Equation 10 eliminates Equation 8 and yields the first and second strain invariant for an incompressible material in Equation 11 and Equation 12.

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \tag{11}$$
$$I_2 = \lambda_1^{-2} + \lambda_2^{-2} + \lambda_3^{-2} \tag{12}$$

Stress-Stretch Relationship for Tension and Compression

For the derivation of the stress-stretch relationships for tension and compression, we consider a cubic differential volume element (Schwarzl 1990, p. 298-299). The volume is subject to the uniaxial tensile stress σ (Figure 1). Axes 1, 2 and 3 denote coordinate axes, which are parallel to the principal axes of the cube. The three principal stretches with regard to the coordinate axes are λ_1, λ_2 and λ_3. If λ is the stretch parallel to the tensile stress σ, deformations in 2 and 3 are equal (Treloar 1973). The corresponding mathematical expressions are

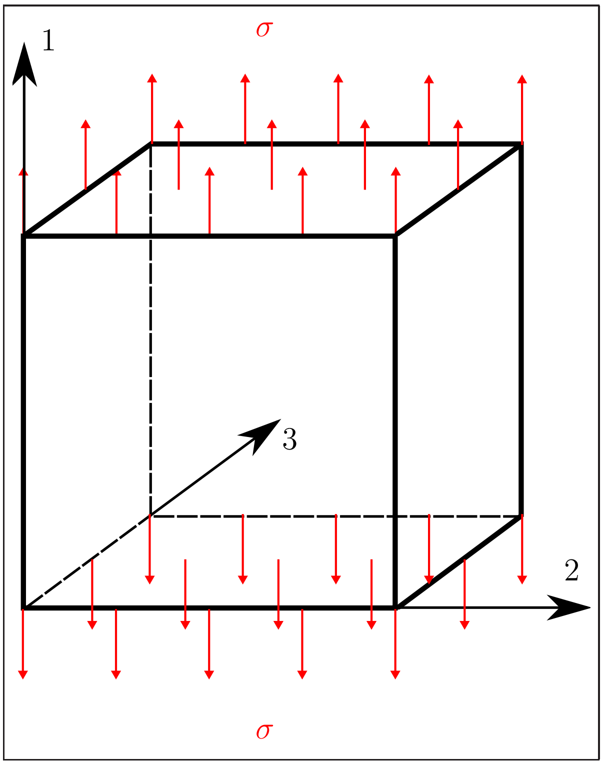

Figure 1
A cubic differential volume element with tensile stress σ and Cartesian coordinate system.

$$\lambda_1 = \lambda \qquad \text{and} \tag{13}$$
$$\lambda_2 = \lambda_3. \tag{14}$$

Since the material is considered incompressible, Equation 10 applies. Together with Equation 14 this yields

$$\lambda_2 = \lambda_3 = \lambda^{\frac{1}{2}}. \tag{15}$$

Equation 13 and Equation 15 can now replace λ_1 and λ_2 in Equation 11 and Equation 12. Resulting from this, the two strain invariants for an incompressible material in tension or compression are

$$I_1 = \lambda^2 + 2\lambda^{-1} \qquad \text{and} \tag{16}$$

$$I_2 = \lambda^{-2} + 2\lambda. \tag{17}$$

The actual relation between engineering stress and stretch for an incompressible material under tension/compression is (Rivlin 1948*b*, 1956)

$$\sigma_e = 2 \cdot \left(\lambda - \lambda^{-2}\right) \cdot \left(\frac{\partial W}{\partial I_1} + \frac{1}{\lambda} \cdot \frac{\partial W}{\partial I_2}\right). \tag{18}$$

Where:

σ_e:	Engineering stress	λ:	Stretch, parallel to σ_e
W:	Strain energy	I_1, I_2:	Strain invariants

Inserting Equation 5 into Equation 18, followed by calculation and simplification, gives

$$\sigma_e = 2 \cdot \left(\lambda - \lambda^{-2}\right) \cdot \Big[C_{10} + 2C_{20}\left(I_1 - 3\right) + \lambda^{-1}C_{01} + 2\lambda^{-1}C_{02}\left(I_2 - 3\right) + C_{11}\left(I_2 - 3 + \lambda^{-1}I_1 - 3\lambda^{-1}\right)\Big]. \tag{19}$$

Finally, using Equation 16 and Equation 17, the two strain invariants I_1 and I_2 can be eliminated. After simplification, the **engineering stress-stretch relationship for tension and compression** is

$$\boxed{\sigma_e\left(\lambda\right) = 2 \cdot \left(\lambda - \lambda^{-2}\right) \cdot \Big[C_{10} + C_{01}\lambda^{-1} + 2C_{20}\left(\lambda^2 + 2\lambda^{-1} - 3\right) + 2C_{02}\left(2\lambda + \lambda^{-2} - 3\right) + 3C_{11}\left(\lambda - 1 - \lambda^{-1} + \lambda^{-2}\right)\Big].} \tag{20}$$

Stress-Strain Relationship for Simple Shear

The derivation of the stress-strain relation for simple shear starts with a differential volume element, as well. It is subject to the shear stress τ (Figure 2). Simple shear strain γ is defined as (Brown 2006, p. 155)

$$\gamma = \frac{x}{h_0}. \tag{21}$$

Where h_0 equals the thickness along axis 2 and x is the displacement along axis 1.

Let λ be the stretch in axis 1. The surfaces parallel to the 1-3 plane move parallel against each other, along axis 1. Therefore the gap distance does not change. The principal stretches λ_1 and λ_2 are (Rivlin 1948*b*, Treloar 1973)

$$\lambda_1 = \lambda \qquad \text{and} \tag{22}$$

$$\lambda_2 = 1. \tag{23}$$

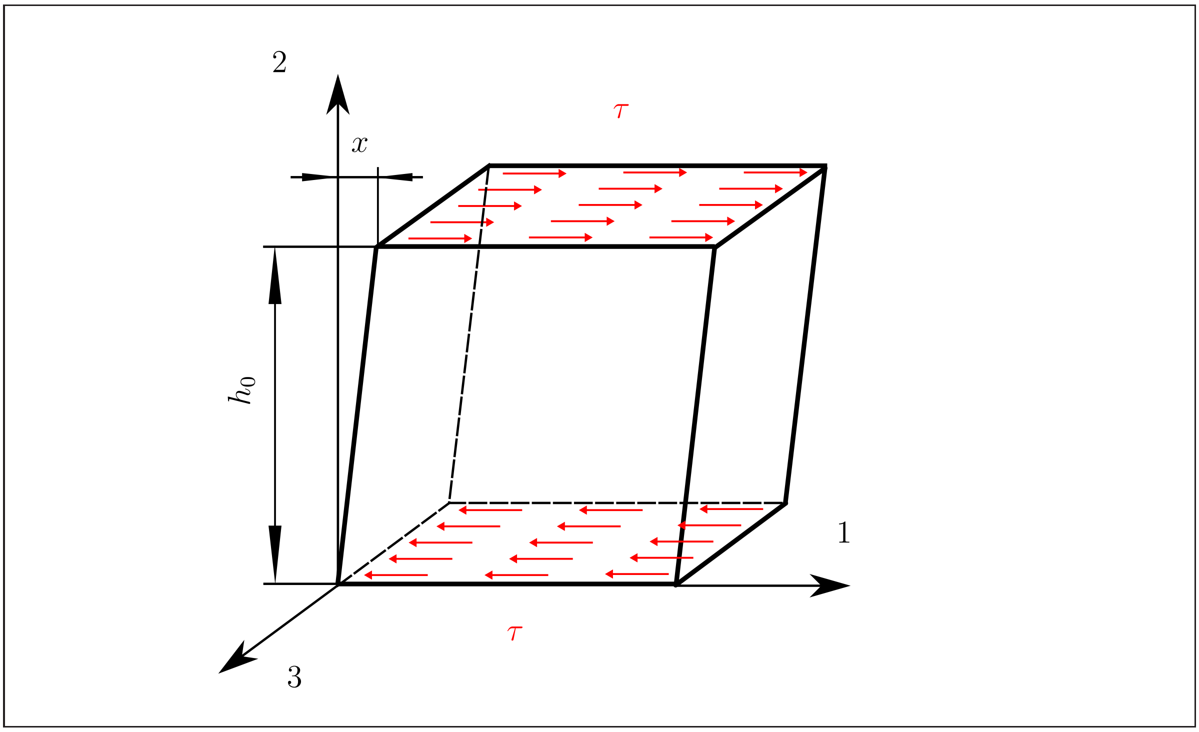

Figure 2
A cubic differential volume element under shear stress τ. The amount of shear strain γ is defined as the quotient of the displacement along axis 1 and the constant height h_0.

Again, assuming incompressibility, Equation 10 applies and λ_3 yields

$$\lambda_3 = \lambda^{-1}. \tag{24}$$

The first and second strain invariant are the same for simple shear and can be expressed in terms of the amount of shear strain γ (Rivlin 1948*b*). They are

$$I_1 = I_2 = 3 + \gamma^2. \tag{25}$$

The stress-strain relationship for simple shear is (Rivlin 1948*a*, 1956)

$$\tau = 2 \cdot \gamma \cdot \left(\frac{\partial W}{\partial I_1} + \frac{\partial W}{\partial I_2} \right). \tag{26}$$

Inserting Equation 5 into Equation 26 gives

$$\tau = 2 \cdot \gamma \cdot \left[C_{01} + C_{10} + 2 \cdot \left(I_1 - 3 \right) \cdot \left(C_{20} + C_{11} + C_{02} \right) \right], \tag{27}$$

which, in combination with Equation 25, yields the **stress-strain relationship for simple shear** in Equation 28.

$$\boxed{\tau\left(\gamma\right) = 2 \cdot \left(C_{01} + C_{10} \right) \cdot \gamma + 4 \cdot \left(C_{20} + C_{11} + C_{02} \right) \cdot \gamma^3} \tag{28}$$

2.2. Definition of the Curve Fitting Problem

Curve Fitting aims to fit one or more parameters of a model equation $\sigma\Big(\lambda, \text{Parameters}\Big)$ in such a way that a given curve $\sigma_M\Big(\lambda\Big)$ is approximated as closely as possible. In this study, the stress-strain curve for simple shear and the combined stress-stretch curve for tension and compression have to be approximated. The stress-stretch curve in tension and compression will be used to explain the regression analysis process. The process for simple shear is analogous.

Ideally, the fitted model equation resulting from regression analysis should yield the same stress values as the measured curve (Equation 29).

$$\sigma\Big(\lambda, \text{Parameters}\Big) = \sigma_M\Big(\lambda\Big) \tag{29}$$

Where $\sigma\Big(\lambda, \text{Parameters}\Big)$ is the model function, which depends on stretch and parameters, and $\sigma_M\Big(\lambda\Big)$ denotes the measured stress-stretch curve.

The model equation for compression and tension was derived in section 2.1 and is Equation 20. It has the form

$$\sigma = f\Big(\lambda, C_{10}, C_{01}, C_{20}, C_{02}, C_{11}\Big). \tag{30}$$

The parameters (material constants) C_{ij} are considered constant and have to be determined through regression analysis. Furthermore, in the compression/tension model equation, they only appear linearly. As the number of data pairs within the measured stress-stretch curve greatly exceeds the number of parameters within the model equation, the problem definition is to solve an overdetermined system of linear equations (Hartmann 2001) in such a way that a satisfactory goodness of fit is achieved.

2.3. Regression Analysis

In general, the overdetermined system of linear equations from subsection 2.2 cannot be solved. In order to overcome this problem, regression analysis can be applied. A set of parameters, which yields to a curve that is as close to the measured curve as possible, has to be determined. For a satisfactory curve fit, the difference between the stress values of the measured curve and the model equation has to be small for a wide rage of stretches.

The least squares method uses the sum of the difference of the ordinates of two stress values as an error criterion (Papula 2008, p. 691), which has to be minimized (Equation 31).

$$\epsilon_{ls} = \sum_{i=1}^{n} \Bigg(\sigma_M\Big(\lambda_i\Big) - \sigma\Big(\lambda_i, C_{10}, C_{01}, C_{20}, C_{02}, C_{11}\Big)\Bigg)^2 \rightarrow \text{minimum} \tag{31}$$

With:

- ϵ_{ls}: Least square error,
- i: Number of measured data pairs,
- λ_i: Measured stretch value,

- $\sigma_M\left(\lambda_i\right)$: Measured stress at λ_i,
- $\sigma\left(\lambda_i, C_{10}, C_{01}, C_{20}, C_{02}, C_{11}\right)$: Computed stress value of the model function at λ_i.

In ANSYS[1], Equation 31 is called "unnormalized least squares fit" ANSYS Inc. (August 2002).

Since Equation 31 is biased towards higher stress values, a weighted error criterion is more useful in many cases. Equation 32 accounts equally for every stress value and is called "normalized least square fit" ANSYS Inc. (August 2002).

$$\epsilon_{norm} = \sum_{i=1}^{n} \left(1 - \frac{\sigma\left(\lambda_i, C_{10}, C_{01}, C_{20}, C_{02}, C_{11}\right)}{\sigma_M\left(\lambda_i\right)} \right)^2 \rightarrow \text{minimum} \tag{32}$$

Note that Equation 32 would lead to a division by zero if $\sigma_M\left(\lambda_i\right) = 0$. For this case an exception has to be added. In this study, Equation 33 was chosen. It is the same formulation as in the least square error in Equation 31. For small values around zero, the normalized and the unnormalized least square criterion yield similar results.

$$\epsilon_{norm} = \left(\sigma_M\left(\lambda_i\right) - \sigma\left(\lambda_i, C_{10}, C_{01}, C_{20}, C_{02}, C_{11}\right) \right)^2 \quad \text{for} \quad \sigma_M\left(\lambda_i\right) = 0 \tag{33}$$

The software ANSYS offers a curve fitting module for hyperelastic material models.[2] However, despite supporting higher order constitutive equations for input of material constants, not all supported material models can be fitted up to these orders. Hence, the Equations 20 and 28 and both error criteria[3] were implemented in a SCILAB[4] script in order to be able to fit higher order constitutive equations. The minimization algorithm used in SCILAB is Nelder-Mead (Nelder & Mead 1965).

Since two curves, one for tension/compression and one for simple shear, have to be fitted in this study, the error used for the minimization process has to consist of the respective errors of both curves. The SCILAB script treats the error for both loading scenarios equally (Equation 34).

$$\epsilon_{min} = \epsilon_{tens/comp} + \epsilon_{shear} \tag{34}$$

With:

- ϵ_{min}: Error used in the minimization process,
- $\epsilon_{tens/comp}$: Error from combined tension and compression curve fitting,
- ϵ_{shear}: Error from simple shear curve fitting.

[1] ANSYS Inc., Canonsburg, Pennsylvania, USA
[2] Simple shear data can only be input in ANSYS Classic.
[3] As in ANSYS, the user can choose which criterion to use.
[4] SCILAB ENTERPRISES, Versailles, France

There exists a global analytic solution for the parameter set in the linear least square problem of the compression/tension curve. However, there is no analytic solution for the nonlinear least square problem that occurs when trying to fit the curve for simple shear (Zielesny 2011, p. 41-42). The method applied in this study is suitable for both linear and nonlinear problem types, and is easily adaptable for different constitutive hyperelastic equations.

As the Ogden models lead to a nonlinear optimization problem which requires an iterative solution, 1000 iterations were carried out for each model in ANSYS, as well as in the Scilab script.

2.4. Evaluation of the Goodness of Fit

The goodness of fit is evaluated using the adjusted coefficient of determination in Equation 35 (Fahrmeir et al. 2009, p. 161). Contrary to the plain coefficient of determination (Equation 36) (Fahrmeir et al. 2009, p. 98-99), it accounts for the number of parameters in the model function. A good fit is indicated if the adjusted coefficient of determination is close to 1, or, ideally, equals 1 for a perfect fit.

$$\bar{R}^2 = 1 - \frac{n-1}{n-p-1} \cdot \left(R^2 - 1\right) \tag{35}$$

$$R^2 = 1 - \frac{\sum\limits_{i=1}^{n} \left(Y_i - \hat{Y}_i\right)^2}{\sum\limits_{i=1}^{n} \left(Y_i - \bar{Y}_i\right)^2} \tag{36}$$

Where:

- $\bar{R}^2$: Adjusted coefficient of determination,
- n: Number of data pairs,
- p: Number of parameters in the model function,
- R^2: Coefficient of determination,
- Y_i: Measured value,
- $\hat{Y}_i$: Value from the model function,
- $\bar{Y}_i$: Average of the measured values.

3. Comparison with a commercial FEA code

Different orders of the polynomial, Ogden and Yeoh model were tested during the curve fitting process. The respective models and their computed material parameters are shown in Tables 1 to 3.

Table 1
Ogden models used in curve fitting and results for the material parameters in ANSYS and using the SCILAB script ($[\mu_i] = \text{MPa}$, $[\alpha_i] = 1$).

Order	error criterion	number of parameters	parameters from curve fitting routine	
			ANSYS	SCILAB script
2nd	norm.	4	$\mu_1 = -0.1244$ $\alpha_1 = 1.316$ $\mu_2 = 0.6225$ $\alpha_2 = 1.299$	$\mu_1 = 4.290$ $\alpha_1 = 0.1370$ $\mu_2 = 7.484 \times 10^{-3}$ $\alpha_2 = 4.346$
3rd	abs.	6	$\mu_1 = 0.1770$ $\alpha_1 = 1.055$ $\mu_2 = 0.2338$ $\alpha_2 = 1.041$ $\mu_3 = 0.3251$ $\alpha_3 = 0.9889$	$\mu_1 = 28.01 \times 10^{-3}$ $\alpha_1 = 3.525$ $\mu_2 = -4.385$ $\alpha_2 = 0.7777$ $\mu_3 = 7.270$ $\alpha_3 = 0.5449$
4th	abs.	8	not available	$\mu_1 = 58.90 \times 10^{-3}$ $\alpha_1 = 2.800$ $\mu_2 = -3.552$ $\alpha_2 = -0.3355$ $\mu_3 = 4.380$ $\alpha_3 = -79.13 \times 10^{-3}$ $\mu_4 = -0.4110$ $\alpha_4 = 0.9359$

Table 2
Polynomial models used in curve fitting and results for the material parameters in ANSYS and using the Scilab script. The model with three parameters is equivalent to a three parameter Mooney-Rivlin model. ($[C_{ij}]$ = MPa).

Order	error criterion	number of parameters	parameters from curve fitting routine: ANSYS	parameters from curve fitting routine: Scilab script
1st	norm.	2	$C_{10} = 37.98 \times 10^{-3}$ $C_{01} = 0.1043$	$C_{10} = 37.98 \times 10^{-3}$ $C_{01} = 0.1043$
-	abs.	3	$C_{10} = 74.92 \times 10^{-3}$ $C_{01} = 73.71 \times 10^{-3}$ $C_{11} = -1.972 \times 10^{-3}$	$C_{10} = 74.92 \times 10^{-3}$ $C_{01} = 73.71 \times 10^{-3}$ $C_{11} = -1.972 \times 10^{-3}$
2nd	abs.	5	$C_{10} = -88.41 \times 10^{-3}$ $C_{01} = 0.2197$ $C_{20} = 0.2122$ $C_{11} = -0.2909$ $C_{02} = 78.22 \times 10^{-3}$	$C_{10} = -88.41 \times 10^{-3}$ $C_{01} = 0.2197$ $C_{20} = 0.2122$ $C_{11} = -0.2909$ $C_{02} = 78.22 \times 10^{-3}$

Table 3
Yeoh models used in curve fitting and results for the material parameters in ANSYS and using the Scilab script ($[C_{i0}]$ = MPa).

Order	error criterion	number of parameters	parameters from curve fitting routine: ANSYS	parameters from curve fitting routine: Scilab script
3rd	norm.	3	$C_{10} = 0.1595$ $C_{20} = -6.930 \times 10^{-3}$ $C_{30} = 0.4030 \times 10^{-3}$	$C_{10} = 0.1595$ $C_{20} = -6.930 \times 10^{-3}$ $C_{30} = 0.4030 \times 10^{-3}$
4rd	norm.	4	not available	$C_{10} = 0.1593$ $C_{20} = -6.220 \times 10^{-3}$ $C_{30} = 0.2210 \times 10^{-3}$ $C_{40} = 11.90 \times 10^{-6}$
5rd	norm.	5	not available	$C_{10} = 0.1573$ $C_{20} = 2.391 \times 10^{-3}$ $C_{30} = -3.907 \times 10^{-3}$ $C_{40} = 0.6600 \times 10^{-3}$ $C_{50} = -32.60 \times 10^{-6}$

4. Discussion

SCILAB scripts were created in order to be able to fit experimental stress-strain data to higher order hyperelastic material models than ANSYS allows for. For lower order models, the scripts were compared against the parameter sets provided by ANSYS.

The SCILAB parameter sets for the polynomial and the Yeoh model are consistent with those from ANSYS (see Table 3 and Table 2). This indicates that the stress-strain equations were derived and implemented correctly, as well as that the regression algorithm and error criteria yield the same results.

However, discrepancies were discovered when fitting the Ogden material models (see Table 1). When examining these differences, a 1st order Ogden model was fitted with absolute and normalized error criteria.[5] For both criteria, the SCILAB script yields similar results to what ANSYS provides. Table 4 shows a comparison of the four parameter sets.

Table 4
Ogden models material parameters, fitted in ANSYS and using the SCILAB script ($[\mu_i] =$ MPa, $[\alpha_i] = 1$).

Order	error criterion	number of parameters	parameters from curve fitting routine: ANSYS	SCILAB script
1st	abs.	2	$\mu_1 = 0.7055$ $\alpha_1 = 1.0546$	$\mu_1 = 0.7208$ $\alpha_1 = 1.0345$
1st	norm.	2	$\mu_1 = 0.5571$ $\alpha_1 = 1.1609$	$\mu_1 = 0.6356$ $\alpha_1 = 1.023$

These data suggest that the discrepancies are not due to false implementation of the stress-strain relationships of the Ogden models, but because of other differences, such as the optimization algorithm used and the nonlinear nature of the problem. In particular, the latter combination and the existence of more than one minimum makes it difficult to effectively compare both curve fitting algorithms for the Ogden model. Even using identical start values for the iteration process may lead to completely different material parameter sets (Zielesny 2011, p. 146). This effect increases with higher order models.

The curve fitting algorithms presented in this study allow fitting of compression/tension and simple shear test data to any order Ogden models, any order Yeoh models and 2nd, as well as 1st, order polynomial models. As a special case of the polynomial model, a three parameter Mooney-Rivlin model may be fitted, too. Contrary to the curve fitting module in ANSYS, the SCILAB scripts allow for biasing of one of the two loading cases (see Equation 34) and increased adjustability of optimization parameters.

In addition to that, auxiliary conditions for the material constants could be considered by extending the source code. However, this was not taken into account within this study in order to be able to compare the results to those from ANSYS. Furthermore, as the

[5] Both of these models were no longer considered after the preselection of material parameter sets, due to a bad fit.

material constants within phenomenological constitutive models do not have a physical meaning, there is only the pursuit of material model stability, which would endorse the introduction of auxiliary conditions. Nonetheless, restricting the Mooney-Rivlin model to positive parameter values, for example, “seems to be too restrictive” (Hartmann 2001). It would also be recommendable to use an error criterion for curve fitting, which takes into account the error in measurement during the curve fitting process (Zielesny 2011, p. 58).

A. Appendix

This appendix contains further information on the derivation of stress-strain relationships for the Ogden and Yeoh material models, as well as the curve fitting results, which were not considered for use within this study.

A.1. Derivation of Stress-Strain Relationships

The derivation of stress-strain relationships for tension/compression and simple shear for the Yeoh as well as the Ogden hyperelastic model is shown in the following. As with the Polynominal model, material incompressibility was considered.

A.1.1. Yeoh Model

The constitutive equation for the Yeoh model is shown in Equation 4 on page 1. The strain energy W is

$$W = \sum_{i=1}^{n} C_i \cdot \Big(I_1 - 3\Big)^i. \tag{37}$$

A.1.1.1. Tension and Compression

If one applies Equation 18 to Equation 37 this yields

$$\sigma_e = \sum_{i=1}^{n} 2 \cdot C_i \cdot i \cdot \Big(\lambda - \lambda^{-2}\Big) \cdot \Big(I_1 - 3\Big)^{i-1} \tag{38}$$

for the tensile/compressive engineering stress.

The first strain invariant in Equation 38 for tension and compression can be replaced by Equation 16, leading to the stress-stretch relation in tension and compression, which is

$$\sigma_e\Big(\lambda\Big) = \sum_{i=1}^{n} 2 \cdot C_i \cdot i \cdot \Big(\lambda - \lambda^{-2}\Big) \cdot \Big(\lambda^2 + 2\lambda^{-1} - 3\Big)^{i-1} \tag{39}$$

for a n^{th} order Yeoh model.

A.1.1.2. Simple Shear

Inserting Equation 37 into Equation 26 gives

$$\tau = \sum_{i=1}^{n} 2 \cdot \gamma \cdot C_i \cdot i \cdot \left(I_1 - 3\right)^{i-1}. \tag{40}$$

If one combines Equation 40 with Equation 25, this yields the stress-strain relation for simple shear of a n^{th} order Yeoh model, which is

$$\tau\left(\gamma\right) = \sum_{i=1}^{n} 2 \cdot \gamma \cdot C_i \cdot i \cdot \gamma^{2\cdot(i-1)}. \tag{41}$$

A.2. Ogden model

The Ogden model is based on principal stretches. Its constitutive equation for the strain energy W is Equation 42 (also see Equation 1).

$$W = \sum_{p=1}^{n} \frac{\mu_p}{\alpha_p} \cdot \left(\lambda_1^{\alpha_p} + \lambda_2^{\alpha_p} + \lambda_3^{\alpha_p} - 3\right) \tag{42}$$

A.2.1. Compression and Tension

In compression and tension, the relations as described in Equation 13 and Equation 15 apply. Inserting them into Equation 42 yields

$$W\left(\lambda\right) = \sum_{p=1}^{n} \frac{\mu_p}{\alpha_p} \left(\lambda^{\alpha_p} + 2 \cdot \lambda^{-\frac{1}{2}\alpha_p} - 3\right). \tag{43}$$

According to Doghri (2000), as cited in Stommel et al. (2012, p. 77), the stress-strain relation for compression and tension can be obtained by deriving the strain energy with respect to the stretch (Equation 44).

$$\sigma\left(\lambda\right) = \frac{\partial W\left(\lambda\right)}{\partial \lambda} \tag{44}$$

Inserting Equation 43 into Equation 44 yields the engineering stress-strain equation for compression and tension for a n^{th} order Ogden model, which is shown in Equation 45.

$$\sigma\left(\lambda\right) = \sum_{p=1}^{n} \mu_p \cdot \left(\lambda^{\alpha_p - 1} - \lambda^{-(\frac{1}{2}\alpha_p + 1)}\right) \tag{45}$$

A.2.1.1. Simple Shear

The relations of the principal stretches in simple shear are shown in Equations 22 to 24. Inserting them into Equation 42 gives

$$W\left(\lambda\right) = \sum_{p=1}^{n} \mu_p \cdot \left(\lambda^{\alpha_i} + \lambda^{-\alpha_i} - 2\right). \tag{46}$$

Again, Equation 44 applies and finally yields to Equation 47 as the simple shear stress-strain relationship for a n^{th} order Ogden model. (Ogden 1972)

$$\tau\Big(\lambda\Big) = \sum_{p=1}^{n} \mu_p \cdot \frac{\lambda^{\alpha_p} - \lambda^{-\alpha}}{\lambda + \lambda^{-1}} \tag{47}$$

The stretch in simple shear was not measured during the material tests. Ogden (1972) showed that the simple shear strain can be expressed in terms of the stretch as follows:

$$\lambda - \lambda^{-1} = \gamma. \tag{48}$$

Using Equation 48, the stretch cannot be directly calculated by solving the equation for λ. However, Equation 48 may be harnessed in order to iteratively compute the amount of stretch from a given amount of shear. The corresponding function code in SCILAB is:

```
// compute stretch from given amount of shear
// Input:  a simple shear strain value     <gamma>
// Output: a stretch value                 <lambda>

function out = ConvToStretch(gamma)
    lambda = 1;                                 // start value for iter.
    while abs((gamma+lambda^(-1)-lambda))>1e-6  // iteration loop
        lambda = gamma + lambda^(-1);
    end
    out = lambda;
endfunction
```

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2 Errata

This is to inform about two flawed equations in the article "Curve Fitting for Ogden, Yeoh and Polynomial Models" by Michael Rackl.

2.1 Equation 15

In equation 15, a minus sign is missing.

$$\lambda_2 = \lambda_3 = \lambda^{-\frac{1}{2}}$$

2.2 Equation 20

Equation 20 is missing a λ^{-1} in the second line's first term.

$$\begin{aligned}\sigma_e(\lambda) = 2 \cdot (\lambda - \lambda^{-2}) \cdot \big[& C_{10} + C_{01}\lambda^{-1} + 2C_{20}(\lambda^2 + 2\lambda^{-1} - 3) \\ & + 2C_{02}\lambda^{-1}(2\lambda + \lambda^{-2} - 3) + 3C_{11}(\lambda - 1 - \lambda^{-1} + \lambda^{-2})\big]\end{aligned}$$

2.3 Impact on the Scilab source code

Both these equations are and were correctly implemented in the Scilab source code at `https://fileexchange.scilab.org/toolboxes/350000`.
Hence, the source code was not affected by the flawed equations mentioned above.

2.4 Acknowledgments

The author would like to acknowledge Sun Ung Park and Klara Loos for pointing him to these issues.